MODULE IV

* **Complex potential function:**

The velocity potential and the stream function are combined in a new function called the complex potential function, by introducing the complex variable:

$z = x + iy$ in cartesian coordinates

$w = f(z) = f(x+iy)$

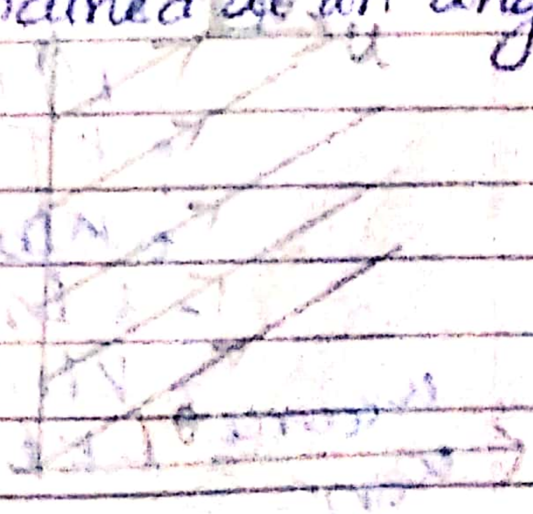
$\Longrightarrow z = re^{i\theta}$ in polar coordinates

$\Longrightarrow w = \phi + i\psi \Longrightarrow$ complex potential function.

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$\frac{dw}{dz} = u - iv$$ $\Longrightarrow$ complex velocity potential function

ii) Uniform flow inclined at an angle 'θ':



$\phi = Ux + Vy$

$\psi = Uy - Vx$

$$\omega = \phi + i\psi$$
$$= (Ux + Vy) + i(Uy - Vx)$$
$$= U(x + iy) + V(y - ix)$$

where $x + iy = z$

$$\omega = Uz + V(y - ix)$$
$$= Uz + -iV(x + iy)$$
$$= Uz + -i \times V(z)$$
$$\omega = Uz - iVz$$
$$\therefore \boxed{\omega = z(U - iV)}$$

For the flow parallel to x axis; $V = 0$

$$\boxed{\omega = zU}$$

For the flow parallel to y axis; $U = 0$

$$\boxed{\omega = -iVz}$$

(ii) Source and sink flow:

$$\omega_{source} = \phi + i\psi$$
$$= K\ln r + iK\theta$$
$$= K[\ln r + i\theta]$$
$$= K[\ln r + \ln e^{i\theta}]$$
$$= K\ln(re^{i\theta})$$
$$\omega_{source} = K\ln(z)$$
$$\omega_{sink} = -K\ln(z)$$

$\phi = \frac{\Lambda}{2\pi}\ln r$

Let $\frac{\Lambda}{2\pi} = K$

$\phi = K\ln r$

$\psi = \frac{\Lambda\theta}{2\pi}$

Let $\frac{\Lambda}{2\pi} = K$

$\psi = K\theta$.

(iii) Vortex flow:

$$\omega_{vortex} = \phi + i\psi$$
$$= C\theta - iC\ln r$$
$$= C[\theta - i\ln r]$$
$$= -iC[\ln r + i\theta]$$
$$= -iC[\ln r + \ln e^{i\theta}]$$
$$= -iC\ln(re^{i\theta})$$
$$= -iC\ln(z)$$
$$\omega_{vortex} = -iC\ln(z)$$

$\phi = \frac{-\Gamma}{2\pi}\theta$

Let $\frac{-\Gamma}{2\pi} = C$

$\phi = C\theta$

$\psi = \frac{\Gamma\ln r}{2\pi}$

$\psi = -C\ln r$

(iv) Doublet flow:

$$\phi = \frac{K}{2\pi}\frac{\cos\theta}{r} \qquad \psi = \frac{-K}{2\pi r}\sin\theta$$

(K = kappa)

$$\Rightarrow w = \phi + i\psi$$
$$\Rightarrow \omega_{doublet} = \frac{K}{2\pi r}\cos\theta - \frac{iK}{2\pi r}\sin\theta$$
$$= \frac{K}{2\pi r}[\cos\theta - i\sin\theta]$$
$$= \frac{K}{2\pi r}[e^{-i\theta}]$$
$$= \frac{K}{2\pi}\frac{1}{re^{i\theta}}$$
$$\omega_{doublet} = \frac{K}{2\pi}\left(\frac{1}{z}\right)$$

(v) Doublet in a uniform flow:

$$\omega = \omega_{uniform} + \omega_{doublet}$$

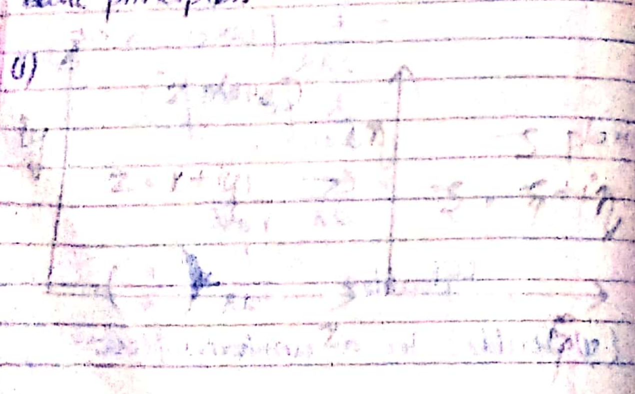
$$\omega = zu + \frac{k}{2\pi}\left(\frac{1}{z}\right)$$

* Conformal Transformation:

It is the study of methods, where by an orthogonal geometric pattern (plane 1) composed of certain shaped elements can be transformed into entirely different (Plane 2) when the elements retain their distinctive form and proportion.

* Basic principles:

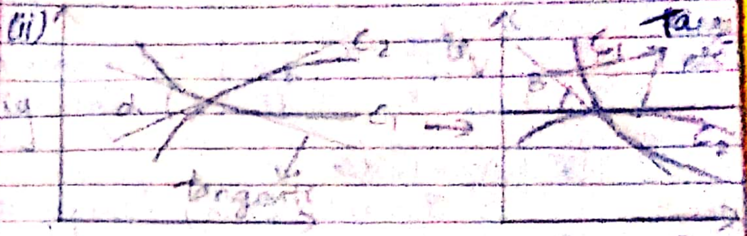
(i)



Here $\zeta = f(z) \rightarrow$ Transformation function

$$\boxed{d\zeta = f'(z)\cdot dz}$$

(ii)



$\boxed{\alpha = \beta} \rightarrow$ Conformal Transformation.

The transformation is said to be conformal, if after being transformed, elements retain their geometric intersection angles.

(iii) Length ratio's between corresponding elements in transformed planes $= \left|\frac{d\zeta}{dz}\right|$.

(iv) Velocity ratio's between corresponding points in transformed planes:

$$\frac{\text{velocity in the transformed plane}}{\text{velocity in original plane}} = \frac{\frac{d\omega}{d\zeta}}{\frac{d\omega}{dz}}$$

ie, $\frac{q_{\xi}}{q_z} = \left|\frac{dz}{d\xi}\right|$

$\longrightarrow$ $\boxed{\frac{q_{\xi}}{q_z} = \frac{1}{\left|\frac{d\xi}{dz}\right|}}$ $\longrightarrow$ Inverse of length ratio.

(4) Singularities:

The relationship between corresponding elements is defined by:

$d\xi = f'(z)dz$

This relation break down when

$f'(z) =$ zero or infinite.

In both cases the conformality of the transformation is lost. The points at which $f'(z) = 0$ or ∞ is known as (in any transformation) singular points, commonly abbreviated as singularities.

* Performing a Transformation:

Expand $\xi = f(z) = \xi + i\eta$ and by equating real and imaginary parts, find functions of ξ and η in terms of x and y.

ie, find $\xi = f_1(x, y)$

$\eta = f_2(x, y)$

Thus any point $P(x, iy)$ gives points $P(\xi, i\eta)$ provided transformation formula $\xi = f(z)$ is given.

* Examples of simple transformation

1) Transformation of a uniform flow parallel to x axis, using the transformation formula

$\xi = z^2$

we have $z = x + iy$.

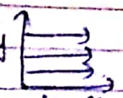
Given that $\xi = z^2$

$= (x + iy)^2$ $\quad [i^2 = -1.]$

$= x^2 - y^2 + i2xy$

Real part $= x^2 - y^2 = \xi$ —— (1)

Imaginary part $= 2xy = \eta$ —— (2)



For uniform flow, Stream function is:

$\psi = Uy$

$y = \frac{\psi}{U} = k$ (constant) —— (3)

from (2), $y = \frac{\eta}{2x}$ —— (4)

Equate (3) & (4):

$$K = \frac{\eta}{2x}$$

ie, $\boxed{x = \frac{\eta}{2K}}$ — (6)

From (1):

$$\xi = x^2 - y^2$$

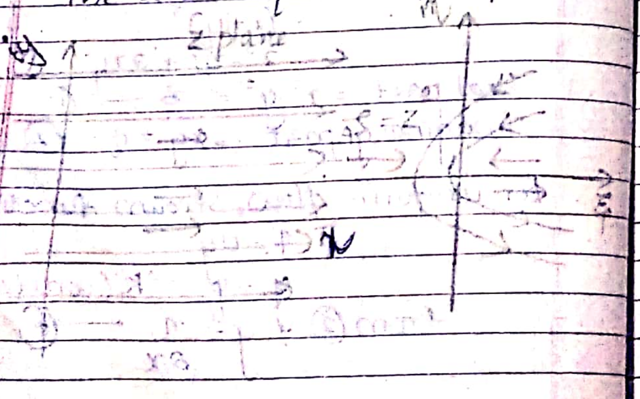
substitute (3) & (6) in (1):

$$\xi = \left(\frac{\eta}{2k}\right)^2 - k^2$$

$$\boxed{\xi = \frac{\eta^2}{4k^2} - k^2}$$

$\boxed{\text{||ar to } x = y^2}$

The above equation is a parabola.



(ii) Transformation of uniform flow parallel to y-axis using $\xi = z^2$.

We have, $z = x + iy$

Given that: $\xi = (x+iy)^2$

$$= (x^2 - y^2) + i2xy.$$

(1) — $\boxed{\xi = x^2 - y^2}$; $\boxed{\eta = 2xy}$ — (2)

For a uniform flow parallel to y axis; stream function is:

$$\psi = Vx$$

$$x = \frac{\psi}{V} = k \text{ constant} \quad \text{— (3)}$$

Also from (2)

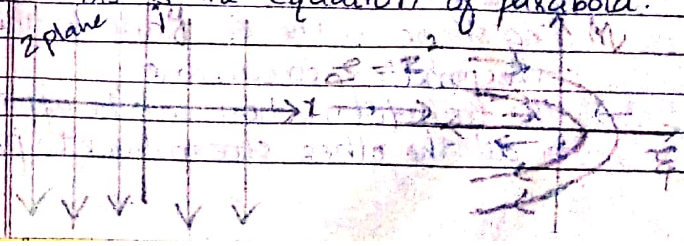
$$y = \frac{\eta}{2x}$$ and substitute (3)

$$\therefore \boxed{y = \frac{\eta}{2k}} \quad \text{— (4)}$$

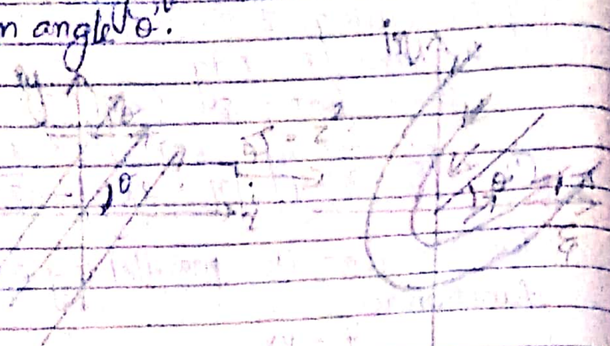
substitute (3) & (4) in (1):

$$\boxed{\xi = k^2 - \frac{\eta^2}{4k^2}}$$

This is the equation of parabola.



(iii) Similarly if uniform flow is inclined at an angle 'θ'.



* Kutta - Joukowski's Transformation:

→ This transformation is the simplest of all other transformations for producing airfoil shaped contours.

→ It confines the study to the actual contour of the circle and show how its shape changes on transformation.

→ The circle is the only one stream line of the flow in z-plane, the corresponding shape in the ζ-plane is the corresponding streamline.

→ The transformation can be applied to all the other streamlines to produce the corresponding streamlines in ζ-plane flowing around an airfoil.

→ The transformation function is given by:

$$\boxed{\zeta = z + \frac{b^2}{z}} \quad \text{where } b \rightarrow \text{constant}$$

$$\zeta = re^{i\theta} + \frac{b^2}{re^{i\theta}}$$

$$= r(\cos\theta + i\sin\theta) + \frac{b^2}{r}(\cos\theta - i\sin\theta)$$

$$= r\cos\theta + \frac{b^2}{r}\cos\theta + i\left[r\sin\theta - \frac{b^2}{r}\sin\theta\right]$$

$$= r\cos\theta\left[1 + \frac{b^2}{r^2}\right] + i r\sin\theta\left[1 - \frac{b^2}{r^2}\right]$$

we know that: $\zeta = \xi + i\eta$

$$\therefore \boxed{\begin{aligned} \xi &= r\cos\theta\left[1 + \frac{b^2}{r^2}\right] \\ \eta &= r\sin\theta\left[1 - \frac{b^2}{r^2}\right] \end{aligned}}$$

* Transformation of a circular cylinder (circle) into a flat plate (straight line):

* The thickness of the flat plate is assumed to be much smaller than the chord

length.

* centre of the circle is at origin.

* $r = a = b$.

we know that

$$\xi_p = r\cos\theta\left[1+\frac{b^2}{r^2}\right] \qquad r=b$$

$$= \underline{2b\cos\theta}$$

$$\eta = r\sin\theta\left[1-\frac{b^2}{r^2}\right] \qquad r=b$$

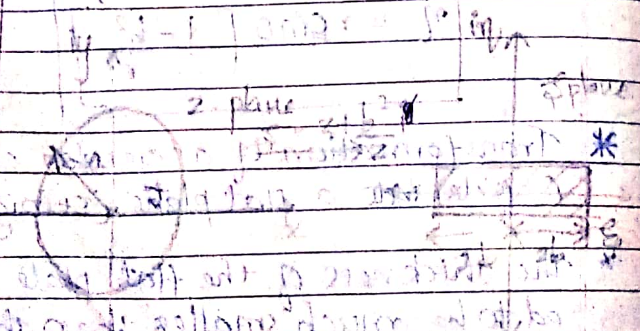
$$= \underline{0}.$$

At leading edge $\theta = 0°$; $\therefore \xi_p = 2b$.

At trailing edge, $\theta = \pi$; $\therefore \xi_p = -2b$

The total length of the plate $= 4b$.

i.e., chord $= 4b$.



(ii) Transformation of a circular cylinder into an elliptical airfoil:

* The centre at origin:

* $r = a$; but $a > b$.

We know that :

$$\xi_p = r\cos\theta\left[1+\frac{b^2}{r^2}\right]$$

$$\xi_p = a\cos\theta\left[1+\frac{b^2}{a^2}\right] \quad \text{—①}$$

$$\eta = r\sin\theta\left[1-\frac{b^2}{r^2}\right]$$

$$\eta = a\sin\theta\left[1-\frac{b^2}{a^2}\right] \quad \text{—②}$$

From ①: $\cos\theta = \dfrac{\xi_p}{a\left[1+\dfrac{b^2}{a^2}\right]}$

From ②: $\sin\theta = \dfrac{\eta}{a\left[1-\dfrac{b^2}{a^2}\right]}$

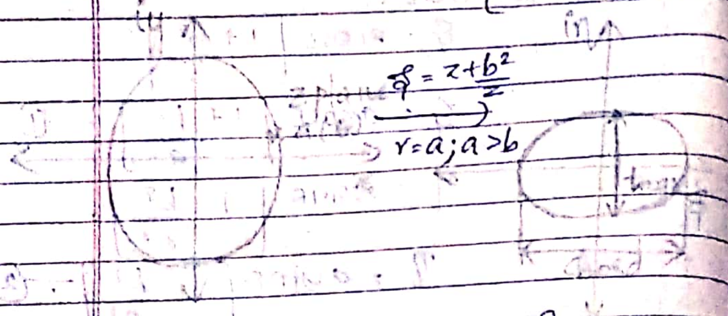
$\cos^2\theta + \sin^2\theta = 1$

i.e., $$\boxed{\frac{\xi_p^2}{a^2\left[1+\frac{b^2}{a^2}\right]^2} + \frac{\eta^2}{a^2\left[1-\frac{b^2}{a^2}\right]^2} = 1.}$$

The above equation is an ellipse with

major axis $= 2a\left[1+\frac{b^2}{a^2}\right]$.

minor axis $= 2a\left[1-\frac{b^2}{a^2}\right]$



$$\text{chord} = 2a\left[1+\frac{b^2}{a^2}\right]$$

$$t_{max} = 2a\left[1-\frac{b^2}{a^2}\right]$$

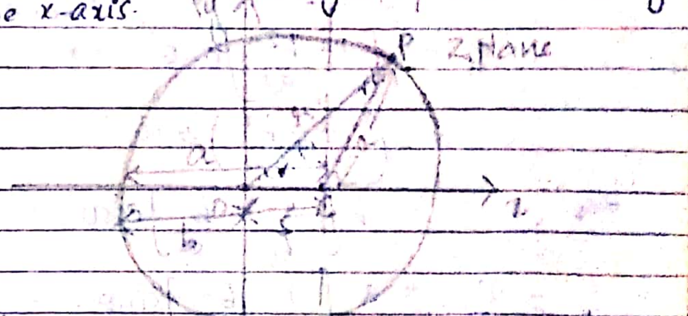
$$\text{finess ratio} = \frac{\text{chord}}{t_{max}} = \frac{2a\left[1+\frac{b^2}{a^2}\right]}{2a\left[1-\frac{b^2}{a^2}\right]} = \frac{a^2+b^2}{a^2-b^2}$$

Thickness to chord ratio $= \frac{t_{max}}{\text{chord}}$

$$= \frac{a^2-b^2}{a^2+b^2} = \frac{1}{\text{finess ratio}}$$

(iii) Transformation of a circular cylinder into a symmetrical airfoil:-

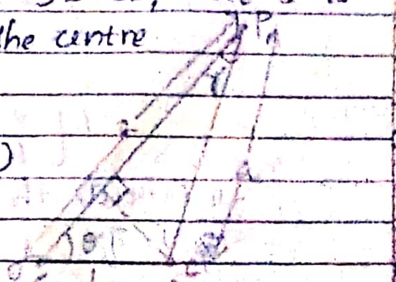
* The circle is not at the origin of z-plane but it is shifted a very small distance along the x-axis.



* Eccentricity $e = \frac{s}{b} \Rightarrow s = eb$, where s is horizontal shift of the centre
* Radius $a = b+s$
$= b+eb$
$= b(1+e)$

From the figure

$r = be\cos\theta + a\cos\gamma$



$1 = be\cos\theta + b(1+e)\cos 1$

for small value of a

$\cos a = 1$

$\therefore a = be\cos\theta + b(1+e)$

$\therefore \boxed{\dfrac{a}{b} = e\cos\theta + 1 + e}$

$\therefore \dfrac{b}{a} = [1 + e + e\cos\theta]^{-1}$

$\therefore \boxed{\dfrac{b}{a} = 1 - e - e\cos\theta}$ [By binomial expansion]

$\xi_p = a\left[1 + \dfrac{b^2}{a^2}\right]\cos\theta$

$= \left(a + \dfrac{ab^2}{a^2}\right)\cos\theta$

$= b\left[\dfrac{a}{b} + \dfrac{b}{a}\right]\cos\theta$

$\eta = a\left[1 - \dfrac{b^2}{a^2}\right]\sin\theta$

$= \left(a - \dfrac{b^2 a}{a^2}\right)\sin\theta$

$= b\left[\dfrac{a}{b} - \dfrac{b}{a}\right]\sin\theta$

substitute the value of $\dfrac{a}{b}$ & $\dfrac{b}{a}$ in ξ_p & η.

$\xi_p = b[1 + e + e\cos\theta + 1 - e - e\cos\theta]\cos\theta$

$\boxed{\xi_p = 2b\cos\theta}$

$\eta = b[1 + e + e\cos\theta - 1 + e + e\cos\theta]\sin\theta$

$= b[2e + 2e\cos\theta]\sin\theta$

$\boxed{\eta = 2be[1 + \cos\theta]\sin\theta}$

Plotting the value of ξ_p and η for different θ, we get symmetrical airfoil in ξ plane.

* Thickness to chord ratio:-

Maximum thickness occurs at

$\dfrac{d\eta}{d\theta} = 0$

$\because \eta = 2be(1+\cos\theta)\sin\theta$

$\dfrac{d\eta}{d\theta} = 2be[(1+\cos\theta)\cos\theta + \sin\theta(-\sin\theta)] = 0$

$= 2be[\cos\theta + \cos^2\theta - \sin^2\theta] = 0$

$\cos^2\theta - \sin^2\theta + \cos\theta = 0$

$\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta - (1-\cos^2\theta) + \cos\theta = 0$

$\cos^2\theta - 1 + \cos^2\theta + \cos\theta = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$2\cos\theta - 1 = 0$; $\cos\theta + 1 = 0$

$\cos\theta = 1/2$; $\cos\theta = -1$

$\therefore \theta = \frac{\pi}{3}$ (Maximum) ; $\therefore \theta = \pi$ (Minimum).

when $\theta = \frac{\pi}{3}$, we get the position of maximum thickness

$\xi_1 = 2b\cos\left(\frac{\pi}{3}\right)$

$\boxed{\xi_{max} = b}$ (from origin)

$\eta = 2be\left[1 + \frac{1}{2}\right] \times \frac{\sqrt{3}}{2}$

$\boxed{\eta_{max} = \frac{3\sqrt{3}}{2} be}$

Maximum thickness $= t_{max} = 2\eta_{max}$

$\boxed{\therefore t_{max} = 3\sqrt{3}\, be}$

Thickness to chord ratio $= \frac{t_{max}}{\text{chord}}$; chord $= 4b$

$= \frac{3\sqrt{3}\, be}{4b}$

$\boxed{\frac{t_{max}}{c} = 1.3e}$

(*) (iv) Transformation of a circular cylinder into a cambered aerofoil (Unsymmetrical)

→ The centre is shifted horizontally and vertically.

→ Horizontal shift ⟹ $s = be$

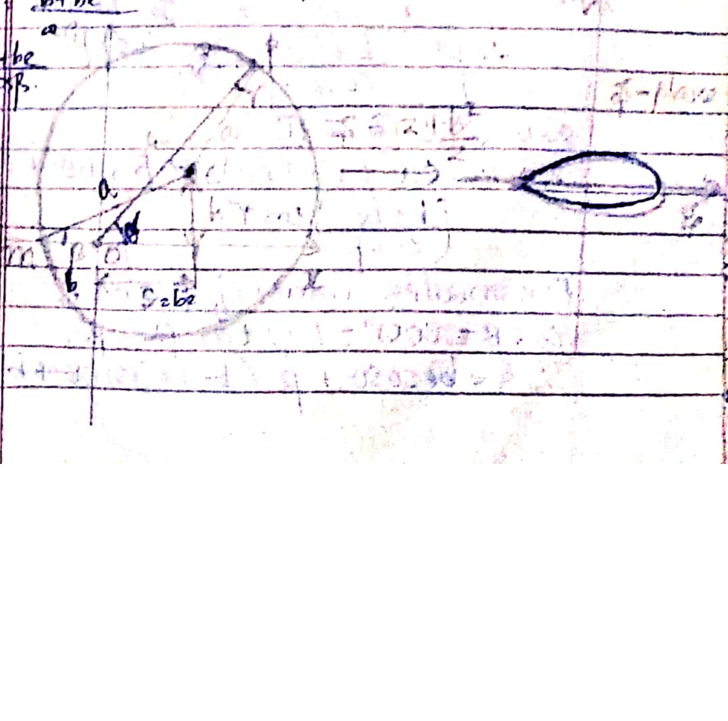
vertical shift ⟹ $h = (b+be)\tan\beta$

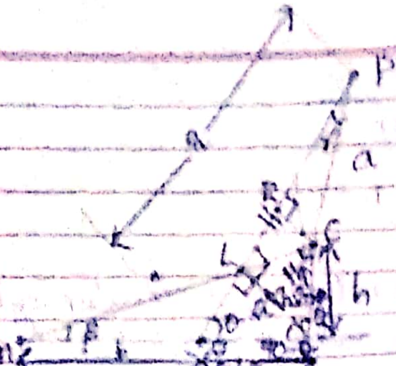
→ The angle γ, β is defined very small.

$\therefore \cos\beta = \cos\gamma = 1$

$a\cos\beta = b + be$

$a = \frac{b+be}{\cos\beta}$





$\begin{bmatrix} OL = be\cos\theta \\ LK = h\sin\theta \\ KP = a\cos\gamma \end{bmatrix}$

$r = OL + LK + KP$

$\boxed{r = be\cos\theta + h\sin\theta + a\cos\gamma}$ — ①

From figure:

$h = (b+be)\tan\beta$ — ②

$CM = CP$ (radius)

$\therefore CM = a;$

$a = CM;\ CM = \dfrac{(b+be)}{\cos\beta}$

$\therefore \boxed{a = \dfrac{b+be}{\cos\beta}}$ — ③

sub ② & ③ in ①:

$r = be\cos\theta + (b+be)\tan\beta\sin\theta + \left(\dfrac{b+be}{\cos\beta}\right)\cos\gamma$

For smaller values of β and γ,

$\cos\beta \cong \cos\gamma \cong 1$; $\therefore \tan\beta = \beta$

$\therefore r = be\cos\theta + \beta(b+be)\sin\theta + b + be$

$= be\cos\theta + \beta b\sin\theta + \beta be\sin\theta + b + be$

$\beta be\sin\theta \approx 0$

$\therefore r = be\cos\theta + \beta b\sin\theta + b + be$

$= b[1 + e + e\cos\theta + \beta\sin\theta]$

$\therefore \boxed{\dfrac{r}{b} = 1 + e + e\cos\theta + \beta\sin\theta}$

$\dfrac{b}{r} = [1 + e + e\cos\theta + \beta\sin\theta]^{-1}$

$\boxed{\dfrac{b}{r} = (1 - e - e\cos\theta - \beta\sin\theta)}$ neglect higher order terms

$\therefore \xi = b\left[\dfrac{r}{b} + \dfrac{b}{r}\right]\cos\theta$

$= b[2\cos\theta] = \underline{\underline{2b\cos\theta}}$

$\therefore \boxed{\xi = 2b\cos\theta}$

$\eta = b\left[\dfrac{r}{b} - \dfrac{b}{r}\right]\sin\theta$

$= b[2e + 2e\cos\theta + 2\beta\sin\theta]\sin\theta$

$\boxed{\eta = 2be[1 + \cos\theta]\sin\theta + 2b\beta\sin^2\theta}$

The term $2b\beta\sin^2\theta$ is called the camber of the airfoil.

* The thickness to chord ratio:

Thickness at any section is given by:

$t = \eta_u - \eta_l$

where η_u and η_l are the vertical ordinates of the upper and lower surfaces.

$t = \eta(\theta) - \eta(-\theta)$

$= 2be(1+\cos\theta)\sin\theta + 2b\beta\sin^2\theta + 2be(1+\cos\theta)\sin\theta - 2b\beta\sin^2\theta$

$\therefore \{ t = 4be(1+\cos\theta)\sin\theta \}$

For maximum thickness:

$\frac{dt}{d\theta} = 0$

$4be\left[(1+\cos\theta)\cos\theta + \sin\theta \times -\sin\theta\right] = 0$

$(1+\cos\theta)\cos\theta - \sin^2\theta = 0$

$(1+\cos\theta)\cos\theta - \left[1-\cos^2\theta\right] = 0$

$\cos\theta + \cos^2\theta - 1 + \cos^2\theta = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\therefore \theta = \frac{\pi}{3} \; ; \; \theta = \pi$

Maximum at $\theta = \frac{\pi}{3}$

substitute the value of θ in the equation of t:

$\therefore \boxed{t_{max} = 3\sqrt{3}\,be}$

* The location of maximum thickness:

$\xi = 2b\cos\frac{\pi}{3}$

$= b$ (from origin).

* Maximum Camber:

$C = 2b\beta\sin^2\theta$

$\frac{dC}{d\theta} = 0$

$2b\beta \times 2\sin\theta \times \cos\theta = 0$

$\sin 2\theta = 0$

$\therefore \theta = \frac{\pi}{2} \; ; \; \theta = 0$

Maximum camber occurs at $\theta = \frac{\pi}{2}$.

$\therefore \boxed{C_{max} = 2b\beta}$

$\Rightarrow \frac{t_{max}}{\text{chord}} = \frac{3\sqrt{3}\,be}{4b}$

$\therefore \boxed{\frac{t_{max}}{c} = 1.3e}$

$\frac{\text{Camber}}{\text{chord}} = \frac{C_{max}}{c} = \frac{2b\beta}{4b} = 0.5\beta$

→ Thin airfoil theory:

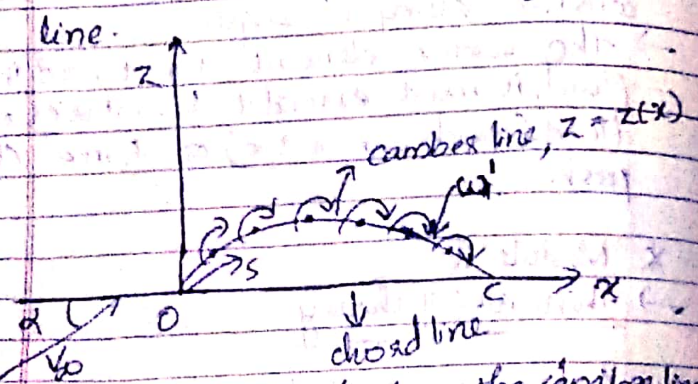
Assumptions made in thin airfoil theory:

(i) Aerofoils is assumed infinitely thin so that it may be represented by its camber line.

(ii) Camber is assumed to be small & the angle of attack is also assumed to be small.

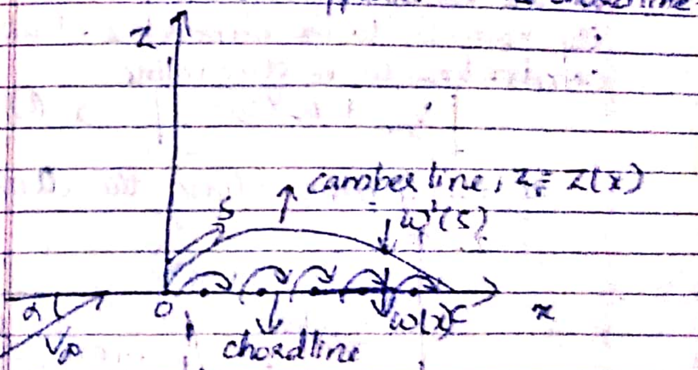
(ii) The flow is assumed to be incompressible and invicid
(iv) The camber line is replaced by vortex sheet of strength 'γ'.

* Classical thin airfoil theory:
The Symmetric Airfoil
The airfoil can be simulated by a vortex sheet placed along the camber line.



vortex sheet on the camber line

The free stream velocity is V_∞ & the airfoil is at an angle of attack 'α'. The x axis is oriented along chord line and the z-axis is ⊥ar to the chord. The distance measured along the camber line is denoted by 's'. The shape of the camber line is $z = z(x)$. The chord length is 'C'. 'w'' is the component of velocity normal to the camber line induced by the vortex sheet. If the airfoil is thin, the camber line is close to the chord line and the vortex sheet appears on the chord line.



vortex sheet on the chordline

$w'(s)$ is the component of velocity normal to the camber line induced by the vortex sheet:

$$w' = w(s)$$

$w(x) \rightarrow$ component of velocity normal to the chord line ; $\boxed{w'(s) = w(x)}$.

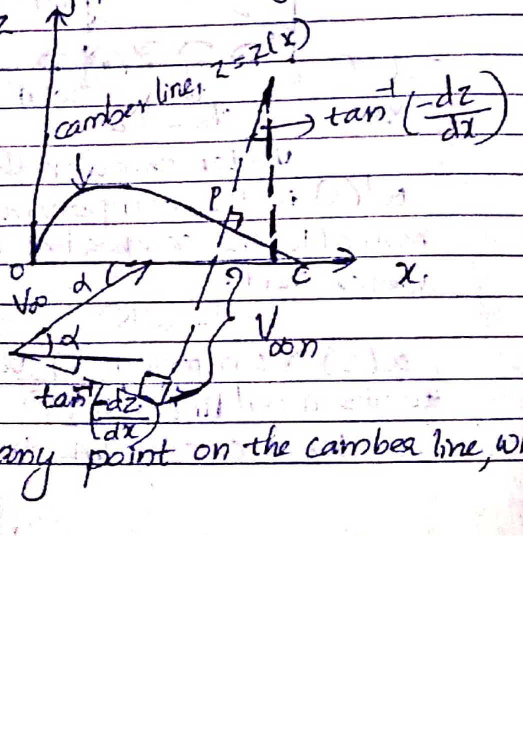
for a camber line to be a stream line,

Scanned with CamScanner

the component of velocity normal to the camberline must be zero at all points along the camber line. The velocity at any point in the flow is the sum of the uniform free stream velocity and velocity induced by the vortex sheet. Let $V_{\infty n}$ be the component of the free stream velocity normal to the camber line. Thus for the camber line to be streamline:

$$\boxed{V_{\infty n} + w'(s) = 0} \longrightarrow ①$$

at every point along the camberline.



At any point on the camber line, where the slope of the camber line is dz/dx.

$$\sin\left[\alpha + \tan^{-1}\left(-\frac{dz}{dx}\right)\right] = \frac{V_{\infty n}}{V_\infty}$$

$$\therefore \boxed{V_{\infty n} = V_\infty \sin\left[\alpha + \tan^{-1}\left(-\frac{dz}{dx}\right)\right]} \longrightarrow ②$$

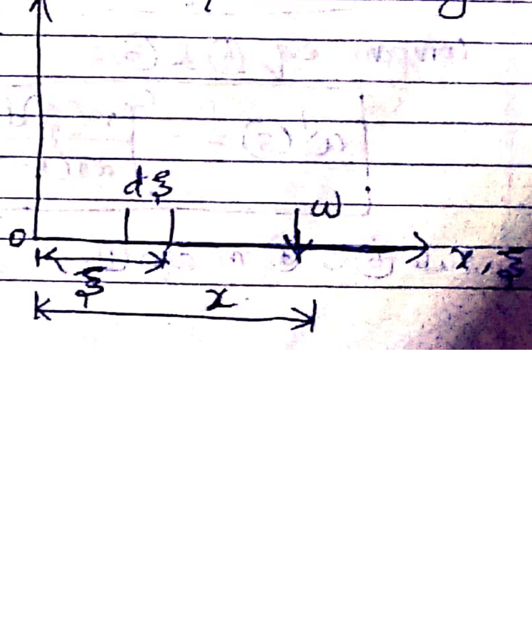
Both α and $\tan^{-1}\left(\frac{dz}{dx}\right)$ are small.

$$\therefore \boxed{V_{\infty n} = V_\infty\left[\alpha - \frac{dz}{dx}\right]} \longrightarrow ③$$

- where α is in radians.

As we know: $\boxed{w'(s) \cong w(x)} \longrightarrow ④$

Consider an elemental vortex of strength 'γ' which varies with the distance along the chord. Consider a section '$d\xi$' at the distance 'ξ' from the origin.



The velocity dw at point x induced by the elemental vortex at point ξ is given by the equation

$$\boxed{dv = -\frac{\Gamma\, ds}{2\pi r}} \rightarrow \text{General form}$$

Illrly in this case

$$\boxed{dw = -\frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}} \quad \text{---} (4)$$

The velocity $w(x)$ induced at point x by all the elemental vortices along the chord line is obtained by integrating the equation from leading edge ($\xi = 0$) to the trailing edge ($\xi = c$).

$$\therefore \boxed{w(x) = -\int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}} \longrightarrow (5)$$

compare eq. (4) & (5):

$$\boxed{w'(s) = -\int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}} \longrightarrow (6)$$

sub (6) & (3) in eqn (1):

$$V_\infty\left[\alpha - \frac{dz}{dx}\right] - \int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)} = 0.$$

$$\boxed{V_\infty\left[\alpha - \frac{dz}{dx}\right] = \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\, d\xi}{(x-\xi)}} \longrightarrow (7)$$

This is the fundamental equation of thin airfoil theory. It is simply a statement that camber line is a streamline of the flow.

Applications: In the case of symmetric airfoil [no camber]. So the camber line is coincident with the chord line, so:

$$\frac{dz}{dx} = 0$$

Eq (7) becomes:

$$\boxed{V_\infty \alpha = \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\, d\xi}{(x-\xi)}} \rightarrow (8)$$

A symmetrical airfoil is treated the same as a flat plate. To integral the eqn (7) & (8) let us transform 'ξ' into 'θ' via the following transformation:

$$\boxed{\xi = \frac{c}{2}(1-\cos\theta)} \;;\; \boxed{d\xi = \frac{c}{2}\sin\theta\, d\theta}$$

(a) ; (b)

Since 'x' is a fixed point, angle corresponds to 'x' is denoted by θ_0.

$$\therefore \boxed{x = \frac{c}{2}\left[1-\cos\theta_0\right]}$$

$\theta = 0$ at leading edge & $\theta = \pi$ at trailing edge. Substitute all these values on eqn. (8):

$$V_\infty \alpha = \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\frac{c}{2}\sin\theta\, d\theta}{\frac{c}{2}(1-\cos\theta_0) - \frac{c}{2}(1-\cos\theta)}$$

$$\boxed{V_\infty \alpha = \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0}} \longrightarrow (9)$$

from Glauret's Hypothesis:

$$\boxed{\gamma(\theta) = 2V_\infty\alpha\frac{\left[1+\cos\theta\right]}{\sin\theta}} \longrightarrow (10)$$

We can verify this solution by sub eq (10) in (9):

$$\boxed{V_\infty \alpha = \frac{1}{2\pi}\int_0^\pi \frac{2V_\infty\alpha\left[1+\cos\theta\right]}{\sin\theta}\frac{\sin\theta\, d\theta}{\cos\theta - \cos\theta_0}}$$

$\longrightarrow (11)$

where RHS $= \frac{V_\infty\alpha}{\pi}\left[\int_0^\pi \frac{1}{\cos\theta - \cos\theta_0}d\theta + \int_0^\pi \frac{\cos\theta}{\cos\theta - \cos\theta_0}d\theta\right] \longrightarrow (12)$

where

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_0} = \frac{\pi\sin n\theta_0}{\sin\theta_0}$$

$$\int_0^\pi \frac{\sin n\theta\sin\theta}{\cos\theta - \cos\theta_0} = -\pi\cos n\theta_0$$

Those are called Glauret's integrals.

So in eq (12) instead of (1), we can sub $\cos\theta$

$$\text{RHS} = \frac{V_\infty\alpha}{\pi}\left[\frac{\pi\sin(0\times\theta_0)}{\sin\theta_0} + \frac{\pi\sin\left[1\times\theta_0\right]}{\sin\theta_0}\right]$$

$$= \frac{V_\infty\alpha}{\pi}\left[\pi\times 0 + \pi\times 1\right]$$

$$= V_\infty\alpha$$

$\therefore \boxed{\text{L.H.S} = \text{R.H.S}} \longrightarrow$ Proved by Glauret's Hypothesis

At trailing edge $\theta = \pi$

Sub. π in eq (10):

$$\therefore \gamma(\pi) = 2V_\infty \alpha \frac{[1+\cos\pi]}{\sin\theta}$$

$$= 0$$

The equation of $\gamma(\theta)$ satisfies by Kutta condition also.

* Lift Equation:

The total circulation around the aerofoil; $\Gamma = \int_0^c \gamma(\xi)\,d\xi$

$$\Gamma = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\,d\theta$$

Sub eq (10):

$$\Gamma = \frac{c}{2}\int_0^\pi \frac{2V_\infty\alpha(1+\cos\theta)}{\sin\theta}\sin\theta\,d\theta$$

$$= c\alpha V_\infty \int_0^\pi (1+\cos\theta)\,d\theta$$

$$= \alpha c V_\infty [\theta + \sin\theta]_0^\pi$$

$$\boxed{\Gamma = \alpha c V_\infty \pi} \longrightarrow (13)$$

Consider Kutta - Joukowski theorem, the lift per unit span, $L' = \rho_\infty V_\infty \Gamma$

$$= \rho_\infty V_\infty \alpha c V_\infty \pi$$

$$\boxed{L' = \pi \rho_\infty V_\infty^2 c\alpha}$$

The lift coefficient, $C_l = \frac{L'}{q_\infty S}$ where $q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$

S = span area

$S = c \times 1$

[∵ unit span area is considered]

$$C_l = \frac{\pi \rho_\infty V_\infty^2 c\alpha}{\frac{1}{2}\rho_\infty V_\infty^2 c}$$

$$\boxed{C_l = 2\pi\alpha} \longrightarrow (C)$$

$$\boxed{\text{Lift slope} = \frac{dC_L}{d\alpha} = 2\pi} :$$

$2\pi\ \text{rad}^{-1}$

or

$0.11\ \text{degree}^{-1}$

The lift coefficient is linearly proportional to angle of attack.

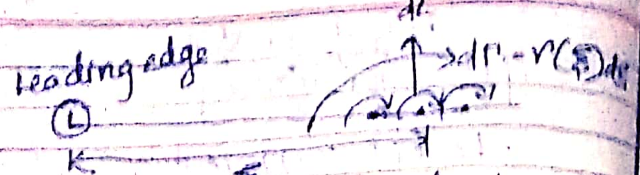
Consider the elemental vortex of strength $\gamma(\xi)d\xi$ located at a distance ξ from the leading edge. The circulation associated with this vortex is $d\Gamma = \gamma(\xi)d\xi$

In turn the increment of lift dL contributed by the elemental vortex is $dL = \rho_\infty V_\infty d\Gamma$.

This increment of lift creates a moment about leading edge.

Figure:

leading edge



Moment about leading edge $dM = -\xi\, dL$

The total moment about the leading edge per unit span due to the entire vortex sheet is:

$$M'_{LE} = -\int_0^c \xi\, dL \quad \text{where } dL = \rho_\infty V_\infty d\Gamma$$

$$M'_{LE} = -\rho_\infty V_\infty \int_0^c \xi\, d\Gamma \quad \text{where } d\Gamma = \gamma(\xi)d\xi$$

$$= -\rho_\infty V_\infty \int_0^c \xi\, \gamma(\xi)\, d\xi$$

Transform the eqn via eqn (a) & eqn (b) & integrate:

$$M'_{LE} = -\rho_\infty V_\infty \int_0^\pi \frac{c}{2}(1-\cos\theta)\, 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta} \times \frac{c}{2}\sin\theta\, d\theta$$

$$= -\rho_\infty V_\infty^2 \frac{\alpha c^2}{2} \int_0^\pi (1-\cos^2\theta)\, d\theta$$

where $\frac{1}{2}\rho_\infty V_\infty^2 = q_\infty$ (Dynamic pressure)

$$= -q_\infty \alpha c^2 \left[\int_0^\pi 1\, d\theta - \int_0^\pi \cos^2\theta\, d\theta\right]$$

$$= -q_\infty \alpha c^2 \left[\pi - \int_0^\pi \frac{1+\cos 2\theta}{2}\, d\theta\right]$$

$$= -q_\infty \alpha c^2 \left[\pi - \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^\pi\right]$$

$$= -q_\infty \alpha c^2 \left[\pi - \left[\frac{\pi}{2}\right]\right]$$

$$= -q_\infty c^2 \alpha \left[\frac{\pi}{2}\right]$$

$$\boxed{\therefore M'_{LE} = -\frac{q_\infty c^2 \alpha \pi}{2}}$$

The moment coefficient:

$$\boxed{C_{M,LE} = \frac{M'_{LE}}{q_\infty S c}} \quad ; S = c \times 1$$

$$= \frac{M'_{LE}}{q_\infty c^2} = \frac{-q_\infty c^2 \alpha \pi}{2 q_\infty c^2} = \frac{-\pi\alpha}{2}$$

$$\boxed{\therefore C_{M,LE} = -\frac{\pi\alpha}{2}}$$

From eq (c):

$C_L = 2\pi\alpha$

$\pi\alpha = \frac{C_L}{2}$

$$\therefore \boxed{C_{M,LE} = -\frac{C_L}{4}}$$

The moment coefficient about the quarter chord point is :

$$\boxed{C_{M,c/4} = C_{M,LE} + \frac{C_L}{4}}$$

$$C_{M,c/4} = -\frac{C_L}{4} + \frac{C_L}{4}$$

$$\therefore \boxed{C_{M,c/4} = 0}$$

This equation demonstrates that the centre of pressure is at quarter-chord point for a symmetarical airfoil. The moment about the quarter chord is zero for all values of α. Hence for a symmetric airfoil, the quarter-chord point is both centre of pressure & the aerodynamic center.

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* Important results:

(i) $C_L = 2\pi\alpha$

(ii) Lift slope $= 2\pi$

(iii) The center of pressure & the aerodynamic center are both located at the quarter-chord point for a symmetrical airfoil.

(iv) $C_M = \frac{-\pi\alpha}{2} = -\frac{C_L}{4}$